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EXACT ASYMPTOTIC BEHAVIOUR OF THE CODIMENSIONS OF SOME P.I. ALGEBRAS

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In memory of S. A. Amitsur, our teacher and friend

ABSTRACT

Let $c_n(A)$ denote the codimensions of a P.I. algebra A, and assume $c_n(A)$ has a polynomial growth: $c_n(A) \underset{n\to\infty}{\simeq} qn^k$. Then, necessarily, $q \in \mathbb{Q}$ [D3]. If $1 \in A$, we show that

$$\frac{1}{k!} \le q \le \frac{1}{2!} - \frac{1}{3!} + - \cdots + \frac{(-1)^k}{k!} \approx \frac{1}{e},$$

where $e = 2.71\ldots$. In the non-unitary case, for any $0 < q \in \mathbb{Q}$, we construct A, with a suitable k, such that $c_n(A) \underset{n\to\infty}{\simeq} qn^k$.

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Introduction

In this paper we consider (non-unitary) P.I. algebras A over a field F of any characteristic. It is known that the codimensions $c_n(A)$ are exponentially bounded [Re], i.e. there exists a constant α such that $c_n(A) \leq \alpha^n$, $n = 1, 2, \ldots$. Our purpose is to obtain more detailed information about the growth of $c_n(A)$. Up till now, all the known examples in characteristic 0 show that either $c_n(A)$ is of polynomial growth or there exist constants a_1, a_2, ℓ_1, ℓ_2 and α such that for all n

$$a_2 n^{\ell_2} \alpha^n \leq c_n(A) \leq a_1 n^{\ell_1} \alpha^n.$$

For a large class of P.I. algebras A, it is now known that $c_n(A)$ asymptotically behaves as

$$c_n(A) \simeq b \cdot n^g \cdot \alpha^n \tag{0.1}$$

for some b, g and α. In all these cases,

$$\alpha \in \mathbb{N}, \tag{0.1.1}$$

$$g \in \frac{1}{2}\mathbb{Z}, \tag{0.1.2}$$

and

$$b = r\left(\frac{1}{\sqrt{\pi}}\right)^s \tag{0.1.3}$$

for some $r \in \mathbb{Q}$ and $0 \leq s \in \mathbb{Z}$ [BR]. It is reasonble to conjecture (0.1) and (0.1.1) in general.

The inverse problem here is that of constructing A with $c_n \simeq b \cdot n^g \cdot \alpha^n$ for given b, g and α. When $\alpha = 1$, this is the case of a polynomial growth of $c_n(A)$. A description of such algebras A (in characteristic 0) was given by Kemer [K] in the language of the cocharacter sequence of A. Further, it is known that

$$c_n(A) = qn^k + \mathcal{O}(n^{k-1}) \simeq qn^k$$

for a rational number q [D3].

The main goal of our paper is to determine the value of q in the case of polynomial growth of $c_n(A)$. It is very surprising that the answer depends on whether or not the algebra is unitary. If $c_n(A) \simeq qn^k$ and $1 \in A$, we show that

$$\frac{1}{k!} \leq q \leq \frac{1}{2!} - \frac{1}{3!} + - \cdots + \frac{(-1)^k}{k!} \approx \frac{1}{e},$$

where $e = 2.71\ldots$. On the other hand, for non-unitary algebras and for any positive rational number q we give explicit constructions of algebras such that $c_n(A)$ is asymptotically equal to qn^k for a suitable k.

1. Unitary algebras

Throughout this paper F is a field of any characteristic and all algebras are F-algebras. We denote by $F\langle X\rangle$ the free non-unitary associative algebra freely generated by a countable set $X = \{x_1, x_2, \ldots\}$ and by $F + F\langle X\rangle$ the free unitary algebra with the same set of free generators. As usual $V_n \subseteq F\langle X\rangle$ is the vector space of the multilinear polynomials in $x_1, \ldots, x_n$. For a P.I. algebra A we denote by $\mathrm{Id}(A)$ the T-ideal of the polynomial identities for A. The sequence $c_n(A) = \dim(V_n/V_n \cap \mathrm{Id}(A))$, $n = 1, 2, \ldots$, is called the codimension sequence of A. Assuming that $c_0(A) = 1$, it is convenient to introduce the generating function

$$c(A,t) = \sum_{n\geq 0} c_n(A)t^n$$

as well as the exponential generating function

$$\tilde{c}(A,t) = \sum_{n\geq 0} c_n(A)\frac{t^n}{n!}.$$

Recall [Sp] that the polynomial $f(x_1, \ldots, x_n) \in V_n$ is called "proper" if it is a linear combination of products of (long) commutators

$$[x_{\sigma(1)}, \ldots] \ldots [\ldots, x_{\sigma(n)}], \quad \sigma \in S_n.$$

We denote by Γ_n the vector space of the proper polynomials of degree n. For a P.I. algebra A we introduce the n-th proper codimension

$$\gamma_n(A) = \dim \Gamma_n/(\Gamma_n \cap \mathrm{Id}(A)), \quad n = 0, 1, 2, \ldots,$$

and the related generating functions

$$\gamma(A,t) = \sum_{n\geq 0} \gamma_n(A)t^n, \quad \tilde{\gamma}(A,t) = \sum_{n\geq 0} \gamma_n(A)\frac{t^n}{n!}.$$

Drensky [D1, D2] has discovered the following relations between the ordinary and the proper codimensions.

PROPOSITION 1.1 ([D1, Corollary 2.5], [D2, p. 322]): *For any unitary algebra A,*

$$c_n(A) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k(A),$$

$$c(A,t) = \frac{1}{t-1}\gamma\left(A, \frac{t}{1-t}\right),$$

$$\tilde{c}(A,t) = e^t \tilde{\gamma}(A,t).$$

In particular, if there exists k such that $\gamma_k(A) \neq 0$ and $\gamma_\ell(A) = 0$ for $\ell > k$, then

$$c_n(A) = \sum_{\ell=0}^{k} \binom{n}{\ell} \gamma_\ell(A)$$

and $c_n(A)$ is a polynomial of degree k in n.

Note that the proofs in [D1, D2] are in characteristic 0. However, they hold without any changes in any characteristic because the result of Specht [Sp] is based on the fact that the free associative algebra is the universal enveloping algebra of the free Lie algebra and this is true over any field.

COROLLARY 1.2 ([Sp]): *For every $n = 0, 1, 2, \ldots$*

$$\dim \Gamma_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right).$$

Proof: Let

$$\tilde{c}(t) = \sum_{n\geq 0} \dim V_n \frac{t^n}{n!} = \sum_{n\geq 0} t^n.$$

$$\tilde{\gamma}(t) = \sum_{k\geq 0} \dim \Gamma_k \frac{t^k}{k!} = \sum_{k\geq 0} \gamma_k \frac{t^k}{k!}.$$

We apply Proposition 1.1 to the free unitary algebra and obtain

$$\tilde{\gamma}(t) = e^{-t}\tilde{c}(t) = \sum_{p\geq 0} \frac{(-1)^p t^p}{p!} \sum_{q\geq 0} t^q = \sum_{n\geq 0} n!\left(\sum_{k=0}^{n} \frac{(-1)^k}{k!}\right)\frac{t^n}{n!},$$

and comparing the coefficients of this series with γ_n we complete the proof. ∎

LEMMA 1.3: *If for a unitary P.I. algebra A there exists k such that $\gamma_{2k}(A) = 0$, then $\gamma_m(A) = 0$ for all $m \geq 2k$.*

Proof: Let $\gamma_{2k}(A) = 0$, i.e. $\Gamma_{2k} \subset \mathrm{Id}(A)$ and let

$$u = [x_{\sigma(1)}, \ldots] \ldots [\ldots, x_{\sigma(n)}] \in \Gamma_n, \quad \sigma \in S_n, \quad n > 2k.$$

If u is a product of commutators of length 2 then n is even and

$$u = ([x_{\sigma(1)}, x_{\sigma(2)}] \cdots [x_{\sigma(2k-1)}, x_{\sigma(2k)}])[x_{\sigma(2k+1)}, x_{\sigma(2k+2)}] \cdots [x_{\sigma(n-1)}, x_{\sigma(n)}]$$

belongs to $\mathrm{Id}(A)$. If u contains a commutator of length greater than 2, e.g.

$$u = [x_{\sigma(1)}, \ldots] \ldots [x_{\sigma(p)}, x_{\sigma(p+1)}, x_{\sigma(p+2)}, \ldots] \ldots [\ldots, x_{\sigma(n)}],$$

then the substitution $y \to [x_{\sigma(p)}, x_{\sigma(p+1)}]$ shows that

$$u = [x_{\sigma(1)}, \ldots] \ldots [[x_{\sigma(p)}, x_{\sigma(p+1)}], x_{\sigma(p+2)}, \ldots] \ldots [\ldots, x_{\sigma(n)}]$$

is a consequence of a commutator from Γ_{n-1}. By inductive arguments we obtain that $u \in \mathrm{Id}(A)$.

Remark: If E is the infinite-dimensional Grassmann algebra, then $c_n(E) = 2^{n-1}$ [KR], and the proof of 1.4(a) below implies that $\gamma_{2k}(E) = 1$ and $\gamma_{2k+1}(E) = 0$ for all k.

THEOREM 1.4: *Let A be a unitary P.I. algebra. Then either*

(a) $c_n(A) \geq 2^{n-1}$ *(hence $c_n(A)$ is exponential) or*

(b) *There exist an integer $k \geq 0$ and a rational number r such that*

$$c_n(A) = r \cdot n^k + \mathcal{O}(n^{k-1}),$$

and

$$\frac{1}{k!} \leq r \leq \left(\frac{1}{e}\right)_k \stackrel{\text{def}}{=} \frac{1}{2!} - \frac{1}{3!} + - \cdots + (-1)^k \frac{1}{k!}.$$

Proof: (a) If $\gamma_{2\ell}(A) \neq 0$ for all $\ell \geq 0$, then by 1.1,

$$c_n(A) = \sum_{j=0}^{n} \binom{n}{j} \gamma_j(A) \geq \sum_{j \geq 0} \binom{n}{2j} = 2^{n-1}.$$

(b) Assume $\gamma_{2\ell}(A) = 0$ for some ℓ. By 1.3, there exists $k \geq 0$ such that $\gamma_k(A) \neq 0$ and $\gamma_m(A) = 0$ for all $m > k$. Thus $c_n(A) = \sum_{j=0}^{k} \binom{n}{j}\gamma_j(A)$, so

$$\binom{n}{k} \leq c_n(A) = \binom{n}{k}\gamma_k(A) + \mathcal{O}(n^{k-1}) = \frac{\gamma_k(A)}{k!}\, n^k + \mathcal{O}(n^{k-1}).$$

The proof follows from 1.2 since $\gamma_k(A)$ is an integer, and $1 \leq \gamma_k(A) \leq \dim \Gamma_k$.

2. Non-unitary algebras with polynomial growth of the codimensions

The purpose of this section is to construct, for any $0 < k \in \mathbb{Z}$, an algebra A such that $c_n(A) \simeq kn^{k-1}$. Given any $0 < q \in \mathbb{Q}$, this allows us to construct (Theorem 3.4 below) an algebra A such that $c_n(A) \simeq qn^p$, for a suitable p.

Fix a positive integer k. Let $M_k(F)$ be the $k \times k$ matrix algebra with entries from F and let $\{e_{pq} \mid p, q = 1, 2, \ldots, k\}$ be the ordinary basis of matrix units for $M_k(F)$, i.e. the only non-zero entry of e_{pq} is 1 in the intersection of the p-th row and the q-th column. Let i be an integer, $1 \leq i \leq k$, and let

$$A_i = Fe_{ii} + \sum_{p<q} Fe_{pq}$$

be the subalgebra of $M_k(F)$ consisting of all upper triangular matrices with all the entries on the diagonal equal to 0 except the (i,i)-entry.

Lemma 2.1: *The algebra A_i satisfies the polynomial identity*

$$f_i(x_1, \ldots, x_{k+1}) = x_1 \ldots x_{i-1}[x_i, x_{i+1}]x_{i+2} \ldots x_{k+1} = 0.$$

Proof: Represent a matrix in A_i by $(e_{ii}, e_{pq} \mid p < q)$ (i.e., a matrix in A_i is a linear combination of these matrix units). Trivially, $[A_i, A_i]$ consists of strictly upper triangular matrices, and these are represented by $(e_{pq} \mid p < q)$ (or (e_{pq}) in short). We need to prove that

$$(*) \qquad \underbrace{(e_{ii}, e_{pq}) \cdots (e_{ii}, e_{pq})}_{i-1}(e_{pq})\underbrace{(e_{ii}, e_{pq}) \cdots (e_{ii}, e_{pq})}_{k-i} = 0.$$

We prove first

Claim: If e_{rs} has a nonzero coefficient in the product

$$\underbrace{(e_{ii}, e_{pq}) \cdots (e_{ii}, e_{pq})}_{i-1}(e_{pq}),$$

then $i+1 \leq s$.

Indeed, e_{rs} can be written as $e_{rs} = e_{r_1 s_1} \cdots e_{r_i s_i}$ with $r_i < s_i$, and for each $1 \leq j \leq i-1$, $e_{r_j s_j}$ either equals e_{ii} or $r_j < s_j$.

CASE 1: For all $1 \leq j \leq i-1$, $e_{r_j s_j} \neq e_{ii}$. It then follows that $r_1 + 1 \leq s_1$, $r_1 + 2 \leq s_2, \ldots, r_1 + i \leq s_i$, so $i+1 \leq s_i$ because $1 \leq r_1$.

CASE 2: The matrix unit e_{ii} appears in that product, so $e_{r,s} = e' e_{ii} e'' e_{r_i s_i}$. Thus $e'' = e_{i r_i}$ with $i \leq r_i < s_i = s$, so, again, $i+1 \leq s$.

We can now prove (*): Assume e_{uv} appears in (*) with a nonzero coefficient, then $e_{uv} = e_{r,s} e_{r_{i+1} s_{i+1}} \cdots e_{r_k s_k}$, and by the above, $i+1 \leq s$. It follows that $e_{r_j s_j} \neq e_{ii}$ (and hence $r_j < s_j$) for all j such that $i+1 \leq j \leq k$. Thus $k+1 = i+1+k-i \leq s+k-i \leq s_k = v$, a contradiction. ∎

Remark: Given n, k and i, $1 \leq i \leq k \leq n$, let

$$L(i,n,k) = \{\sigma \in S_n \mid \sigma(i) < \sigma(i+1) < \cdots < \sigma(n-k+i)\}.$$

Then

$$|L(i,n,k)| = \prod_{j=n-k+2}^{n} j = \binom{n}{k-1} \cdot (k-1)! = n^{k-1} + \mathcal{O}(n^{k-2}).$$

Indeed, $\sigma \in L(i,n,k)$ is completely determined by first choosing $k-1$ values from $\{1,\ldots,n\}$, then ordering them as

$$\sigma(1),\ldots,\sigma(i-1),\sigma(n-k+i+1),\ldots,\sigma(n).$$

Now let $1 \leq i < j \leq k \leq n$, then $L(i,n,k) \cap L(j,n,k) = L(i,n,\bar{k})$, where $\bar{k} = k-j+i \leq k-1$, hence $|L(i,n,k) \cap L(j,n,k)| \simeq n^{\bar{k}-1} = \mathcal{O}(n^{k-2})$. By "the principle of inclusion-exclusion" of Combinatorics,

$$\begin{aligned}\left|\bigcup_{i=1}^{k} L(i,n,k)\right| &\geq \sum_{i=1}^{k} |L(i,n,k)| - \sum_{1 \leq i \neq j \leq k} |L(i,n,k) \cap L(j,n,k)| \\ &\simeq kn^{k-1} + \mathcal{O}(n^{k-2}) \\ &\simeq kn^{k-1}.\end{aligned}$$

Let $1 \leq i \leq k \leq n$ and $1 \leq \ell \leq n$. Denote

$$L(i,n,k,\ell) = \{\sigma \in L(i,n,k) \mid \sigma(n-k+i+1) < n-\ell\},$$

$L'(i,n,k,\ell) = L(i,n,k)\backslash L(i,n,k,\ell)$ if $1 \le i \le k-1$, and $L(k,n,k,\ell) = L(k,n,k)$. Also, let

$$L(n,k) = \bigcup_{i=1}^{k} L(i,n,k,2k-3).$$

Then

LEMMA 2.2:

$$|L(n,k)| \underset{n\to\infty}{\simeq} kn^{k-1} \quad (\textit{In fact, } |L(n,k)| = kn^{k-1} + \mathcal{O}(n^{k-2}).)$$

Proof: Clearly, $L_1 \supseteq L(n,k) \supseteq L_1\backslash L_2$, where $L_1 = \bigcup_{i=1}^{k} L(i,n,k)$ and $L_2 = \bigcup_{i=1}^{k} L'(i,n,k,2k-3)$. By the above, $|L_1| \simeq kn^{k-1}$, hence it suffices to show that for each $1 \le i \le k$ and any ℓ, $|L'(i,n,k,\ell)| = \mathcal{O}(n^{k-2})$. This follows since $L'(k,n,k,\ell) = \emptyset$, and for $1 \le i \le k-1$,

$$\begin{aligned} |L'(i,n,k,\ell)| &= (\ell+1)|L(i,n-1,k-1)| \\ &\simeq (\ell+1)(n-1)^{k-2} \\ &= \mathcal{O}(n^{k-2}). \end{aligned}$$

LEMMA 2.3: *Let $A = A_1 \oplus \cdots \oplus A_k$, A_i, $i = 1,\ldots,k$, as above. Then*

(a) *The monomials $\{M_\sigma(x_1,\ldots,x_n) \mid \sigma \in L(i,n,k)\}$ are linearly independent modulo* $\mathrm{Id}(A_i)$, $i = 1,\ldots,k$.

(b) *The monomials $\{M_\sigma(x_1,\ldots,x_n) \mid \sigma \in L(n,k)\}$ are linearly independent modulo* $\mathrm{Id}(A)$. *(Here $\sigma \in S_n$ and $M_\sigma(x_1,\ldots,x_n) = x_{\sigma(1)}\cdots x_{\sigma(n)}$.)*

Note: From 2.3 (a) and [OR] it follows that all the identities of A_i are consequences of $x_1\ldots x_i[x_i,x_{i+1}]x_{i+2}\ldots x_{k+1}$.

Proof: We prove (b). The proof of (a) is similar — but easier — and is contained in "Case 1" below.

Assume that $g(x_1,\ldots,x_n) = \sum_{\sigma\in L(n,k)} a_\sigma \cdot M_\sigma(x_1,\ldots,x_n) \in \mathrm{Id}(A)$. We denote $L^*(n,k) = \{\sigma \in L(n,k) \mid a_\sigma \ne 0\}$ and proceed to show that $L^*(n,k) = \emptyset$.

Assume that $L^*(n,k) \ne \emptyset$. Denote $\underline{a}_t = (a_{1t},\ldots,a_{kt}) \in A_1 \oplus\cdots\oplus A_k$, $1 \le t \le n$. Since $g(\underline{a}_1,\ldots,\underline{a}_n) = (g(a_{11},\ldots,a_{1n}),\ldots,g(a_{k1},\ldots,a_{kn}))$, it suffices to show that there exist $1 \le j \le k$ and $\overline{x}_1,\ldots,\overline{x}_n \in A_j$ such that $g(\overline{x}_1,\ldots,\overline{x}_n) \ne 0$. So assume $g(\overline{x}_1,\ldots,\overline{x}_n) = 0$ for any $\overline{x}_1,\ldots,\overline{x}_n \in A_j$, $1 \le j \le k$.

Let $j = \min\{i \mid L^*(n,k) \cap L(i,n,k) \ne \emptyset\}$ and let $\tau \in L(j,n,k)$ and $a_\tau \ne 0$. Substitute

$$\begin{aligned} &(\overline{x}_{\tau(1)}, \ldots, \overline{x}_{\tau(j-1)}, \overline{x}_{\tau(j)}, \ldots, \overline{x}_{\tau(n-k+j)}, \overline{x}_{\tau(n-k+j+1)}, \ldots, \overline{x}_{\tau(n)}) \\ &= (e_{1,2}, \ldots, e_{j-1,j}, e_{j,j}, \ldots, e_{j,j}, e_{j,j+1}, \ldots, e_{k-1,k}). \end{aligned}$$

Notice that $\overline{x}_1, \ldots, \overline{x}_n \in A_j$. Clearly,

$$0 = g(\overline{x}) = a_\tau e_{1k} + \sum_{\tau \neq \sigma \in L^*(n,k)} a_\sigma \cdot M_\sigma(\overline{x}_1, \ldots, \overline{x}_n).$$

If $a_\sigma M_\sigma(\overline{x}) = 0$ for all $\tau \neq \sigma \in L^*(n,k)$, then $a_\tau e_{1k} = 0$, so $a_\tau = 0$, a contradiction. Assume $a_\sigma M_\sigma(\overline{x}) \neq 0$ for some $\tau \neq \sigma \in L^*(n,k)$. Then $M_\sigma(\overline{x}) \neq 0$, and it follows that $\sigma(s) = \tau(s)$ for $s = 1, \ldots, j-1, n-k+j+1, \ldots, n$. Since $\sigma \in \bigcup_{i=1}^k L(i,n,k)$, there exists a minimal i ($1 \leq i \leq k$) such that $\sigma \in L(i,n,k)$. By the minimality of j, $j \leq i$.

CASE 1: $i = j$. Then $\sigma(j) < \cdots < \sigma(n-k+j)$, and since these numbers are a permutation of $\tau(j) < \cdots < \tau(n-k+j)$, hence $\sigma(s) = \tau(s)$ also for $s = j, \ldots, n-k+j$. Thus $\sigma = \tau$, a contradiction.

CASE 2: $j + 1 \leq i$. Hence

$$\sigma(n-k+j) < \sigma(n-k+j+1),$$

so $\sigma(i) < \cdots < \sigma(n-k+j+1) = \tau(n-k+j+1)$. Thus, $\tau(n-k+j+1)$ is an upper bound for an increasing sequence of length $n-k+j-i+2 \geq n-2k+3$. Hence

$$\tau(n-k+j+1) \geq n-2k+3,$$

so $\tau \notin L^*(n,k)$, again a contradiction. ∎

We can now prove

THEOREM 2.4: *Let $A = A_1 \oplus \cdots \oplus A_k$ as above, and let $k \leq n$. Then*
(a) *$c_n(A_i) = |L(i,n,k)| = n^{k-1} + \mathcal{O}(n^{k-2}) \simeq n^{k-1}$, and*
(b) *$c_n(A) \simeq kn^{k-1}$. (In fact, $c_n(A) = kn^{k-1} + \mathcal{O}(n^{k-2})$.)*

Proof: (a) By 2.3(a), $c_n(A_i) \geq |L(i,n,k)|$. By 2.1 and by [OR, Th. 3.1] (or by an easy direct argument), $c_n(A_i) \leq n(n-1)\cdots(n-k+2) = |L(i,n,k)|$.

(b) By 2.2 and 2.3,

$$c_n(A) \geq |L(n,k)| \simeq kn^{k-1}.$$

The opposite inequality follows easily: $\mathrm{Id}(A) = \mathrm{Id}(\bigoplus A_i) = \bigcap \mathrm{Id}(A_i)$, hence

$$\frac{V_n}{V_n \cap \mathrm{Id}(A)} = \frac{V_n}{\bigcap (V_n \cap \mathrm{Id}(A_i))} \quad \text{imbeds naturally into } \bigoplus \frac{V_n}{V_n \cap \mathrm{Id}(A_i)}.$$

Thus, by (a),

$$c_n(A) \leq \sum_{i=1}^{k} c_n(A_i) = kn^{k-1} + \mathcal{O}(n^{k-2}). \qquad \blacksquare$$

3. Applications

The main tool for applications here is Theorem 1.4 in [BR], which is a consequence of a theorem of Formanek, and which we now reproduce.

THEOREM 3.1 ([BR, 1.4]): *Let $c_n(A)$ denote the codimensions of a P.I. algebra A. For $j = 1, \ldots, k$, let I_j be T ideals, $I_j = \mathrm{Id}(A_j) \subseteq F\langle X\rangle$, such that $c_n(A_j) \simeq a_j n^{e_j} \alpha_j^n$, $\alpha_1, \ldots, \alpha_k \geq 1$. Let $I = I_1 \cdots I_k$ and let A satisfy $I = \mathrm{Id}(A)$. Then $c_n(A) \simeq an^e\alpha^n$, where $\alpha = \alpha_1 + \cdots + \alpha_k$, $e = e_1 + \cdots + e_k + k - 1$ and*

$$a = a_1 \cdots a_k \frac{\alpha^{e_1} \cdots \alpha_k^{e_k}}{(\alpha_1 + \cdots + \alpha_k)^e}.$$

We also need the following variant of that theorem:

THEOREM 3.2: *Let B_1 be a nilpotent algebra of class $\ell + 1$: $c_\ell(B_1) \neq 0$, $c_{\ell+i}(B_1) = 0$, $i = 1, 2, \ldots$. Denote $c_\ell(B_1) = p_\ell$. Let B_2 be a P.I. algebra such that $c_n(B_2) \underset{n\to\infty}{\simeq} an^e\alpha^n$ ($a > 0$, $\alpha \geq 1$). Let B be a P.I. algebra such that $\mathrm{Id}(B) = \mathrm{Id}(B_1) \cdot \mathrm{Id}(B_2)$. Then*

$$c_n(B) \simeq \frac{p_\ell \cdot a}{\ell! \alpha^{\ell+1}} \cdot n^{\ell+e+1} \cdot \alpha^n.$$

Proof: Denote $w_n = \sum_{j=0}^{n} \binom{n}{j} c_j(B_1) \cdot c_{n-j}(B_2)$. It follows from a formula of Formanek (see [BR, 1.1] for details) that $c_n(B) = c_n(B_1) + c_n(B_2) + nw_{n-1} - w_n$.

Thus, the proof of 3.1 obviously follows from the following

LEMMA 3.3: *Let $\{p_n\}$, $\{q_n\}$ satisfy*

(1) *For some ℓ, $p_\ell \neq 0$ and $p_{\ell+i} = 0$, $i = 1, 2, \ldots$,*

(2) *$q_n \underset{n\to\infty}{\simeq} an^e\alpha^n$, $a > 0$, $\alpha \geq 1$.*

Define $w_n = \sum_{j=0}^{n} \binom{n}{j} p_j q_{n-j}$ and $r_n = p_n + q_n + nw_{n-1} - w_n$. Then

$$r_n \simeq \frac{p_\ell \cdot a}{\ell! \alpha^{\ell+1}} \cdot n^{\ell+e+1} \cdot \alpha^n.$$

Proof: We have

$$\begin{aligned} w_n &= \sum_{j=0}^{\ell} \binom{n}{j} p_j q_{n-j} \\ &= \binom{n}{\ell} p_\ell q_{n-\ell} + \sum_{j=0}^{\ell-1} \binom{n}{j} p_j q_{n-j} \\ &\simeq p_\ell \frac{n^\ell}{\ell!} \cdot a(n-\ell)^e \cdot \alpha^{n-\ell} + \sum_{j=0}^{\ell-1} \binom{n}{j} p_j \frac{n^j}{j!} a(n-j)^e \cdot \alpha^{n-j}. \end{aligned}$$

Now $n - \ell \simeq n \simeq n - j$, hence the first summand dominates the sum, so

$$w_n \simeq \frac{p_\ell \cdot a}{\ell! \alpha^k} \cdot n^{\ell+e} \cdot \alpha^n.$$

It clearly follows that nw_{n-1} dominates r_n, hence

$$\begin{aligned} r_n &\simeq nw_{n-1} \\ &\simeq n \cdot \frac{p_\ell \cdot a}{\ell! \alpha^\ell}(n-1)^{\ell+e} \alpha^{n-1} \\ &\simeq \frac{p_\ell \cdot a}{\ell! \alpha^{\ell+1}} \cdot n^{\ell+e+1} \cdot \alpha^n. \qquad ∎ \end{aligned}$$

We can now prove

THEOREM 3.4: *Let q be an arbitrary positive rational number. Then there exists a (non-unitary) P.I. algebra B such that $c_n(B) \simeq qn^p$, for a suitable positive integer p.*

Proof: Let $q = u/v$, u, v be positive integers. Construct a commutative algebra B_1 which is nilpotent of class $v + 1$, e.g. B_1 has basis $t, t^2, \ldots, t^v$, and $t^{v+1} = 0$. Thus $c_j(B_1) = 1$ if $1 \le j \le v$ and $c_j(B_1) = 0$ if $v < j$.

Denote $k = u \cdot ((v-1)!)$ and let $B_2 = A = A_1 \oplus \cdots \oplus A_k$ as in 2.3:

$$c_n(B_2) \simeq kn^{k-1}.$$

Let B be a P.I. algebra with T-ideal of identities

$$\mathrm{Id}(B_3) = \mathrm{Id}(B_1) \cdot \mathrm{Id}(B_2).$$

Applying 3.1, we obtain ($a = k$, $e = k - 1$, $\alpha = 1$, $\ell = v$, and $p_\ell = 1$):

$$c_n(B_3) \simeq \frac{k}{v!} n^{v+k} = qn^{v+k}. \qquad ∎$$

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